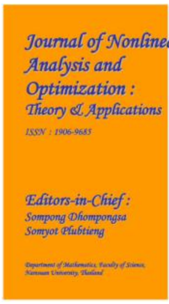

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SOLVING DIOPHANTINE EQUATIONS USING METALLIC RATIOS

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Abstract
Diophantine Equations has been of special interest to many mathematicians for several centuries. Equations whose solutions are expected to be in integers are usually termed as Diophantine equations. Thousands of equations of Diophantine type has been solved by several mathematicians. Some equations have created a great legacy like Fermat's Last Theorem. Among several special types of real numbers that exist, metallic ratios occupy special interest since they possess innumerable mathematical properties. In this paper, we will introduce a pair of quadratic Diophantine Equations and obtain its solutions using the continued fraction expansion of metallic ratios of order *k*. This concept will not only be mathematically pleasing but provides various scopes for solving similar equations.

Keywords: Diophantine Equations, Metallic Ratios of order *k*, Continued Fraction, Convergents, Recurrence Relations

1. Introduction

Diophantine Equations named after second century BCE Greek mathematician Diophantus are very special equations because such equations require solutions to be in integers. Equations whose solutions are integers found abundant applications to practical life problems right from ancient times to modern times. In this sense, galaxy of mathematicians throughout the globe solved various types of Diophantine equations, some of which had created great impact upon development of mathematics. Carl Friedrich Gauss, considered to be one of the great mathematicians of all times, had provided clues to solve quadratic Diophantine equations in his classic book Disquisitiones Arithmeticae. This book created a huge sensation among later mathematicians and paved way for enormous development in number theory. In this paper, we will introduce pair of quadratic Diophantine equations and try to solve them using special class of numbers called metallic ratios in a novel way using continued fraction expansion.

2. Definition

The sequence of metallic ratios of order *k*, denoted by M_k is defined as $M_k = \frac{k+\sqrt{k^2+4}}{2}$ (1). If k = 1, $M_1 = \frac{1+\sqrt{5}}{2}$ is called the famous Golden Ratio

If $k = 2$, $M_2 = 1+\sqrt{2}$ is called the Silver Ratio

If $k = 3$, $M_3 = \frac{3+\sqrt{13}}{2}$ is called the Bronze Ratio

To know more about Metallic Ratios and their properties see **[1 – 4]**.

3. Describing the Equations

The main purpose of this paper is to solve pair of equations $x^2 - kxy - y^2 = \pm 1$ (2) in positive integers, where k is any positive integer. In order to solve pair of quadratic Diophantine equations (2), we first derive the continued fraction expansion of the metallic ratio of order k as defined in (1).

We begin our exploration by noticing that $\left(\frac{\sqrt{k^2+4}-k}{2}\right)\times\left(\frac{\sqrt{k^2+4}+k}{2}\right)=1$

$$\frac{\sqrt{k^2+4}-k}{2} = \frac{1}{\frac{\sqrt{k^2+4}+k}{2}} = \frac{1}{k+\left(\frac{\sqrt{k^2+4}-k}{2}\right)}$$

$$= \frac{1}{k+\frac{1}{k+\left(\frac{\sqrt{k^2+4}-k}{2}\right)}} = \cdots = \frac{1}{k+\frac{1}{k+\frac{1}{k+\frac{1}{k+\frac{1}{k+\cdots}}}}}$$

Thus, $M_k = \frac{k+\sqrt{k^2+4}}{2} = k + \left(\frac{\sqrt{k^2+4}-k}{2}\right) = k + \frac{1}{k+\frac{1}{k+\frac{1}{k+\frac{1}{k+\frac{1}{k+\cdots}}}}} = \left[k;\overline{k}\right]$ (3)

Equation (3) provides the continued fraction expansion of the metallic ratio of order k.

Let us now consider the consecutive convergents of the continued fraction expansion of metallic ratio of order k using (3). Doing this we obtain,

$$\left.\begin{aligned} &c_0 = \frac{k}{1}, c_1 = k + \frac{1}{k} = \frac{k^2+1}{k}, c_2 = k + \frac{k}{k^2+1} = \frac{k^3+2k}{k^2+1}, c_3 = k + \frac{k^2+1}{k^3+2k} = \frac{k^4+3k^2+1}{k^3+2k}, \\ &c_4 = k + \frac{k^3+2k}{k^4+3k^2+1} = \frac{k^5+4k^3+3k}{k^4+3k^2+1}, c_5 = k + \frac{k^4+3k^2+1}{k^5+4k^3+3k} = \frac{k^6+5k^4+6k^2+1}{k^5+4k^3+3k}, \cdots \end{aligned}\right\} \quad (4)$$

Now considering the numerators and denominators of rational numbers in the successive convergents of (4) as x and y respectively, we notice that $c_0, c_2, c_4, ...$ form solutions to $x^2 - kxy - y^2 = -1$ and $c_1, c_3, c_5, ...$ provide solutions to $x^2 - kxy - y^2 = 1$ for any positive integer k.

In particular, $(x, y) = (k,1); (k^3+2k, k^2+1); (k^5+4k^3+3k, k^4+3k^2+1); ...$ (5) provides solutions to $x^2 - kxy - y^2 = -1$ and

$(x, y) = (k^2+1, k); (k^4+3k^2+1, k^3+2k); (k^6+5k^4+6k^2+1, k^5+4k^3+3k); ...$ (6) provides solutions to $x^2 - kxy - y^2 = 1$.

Since, there are infinite convergents that can be generated from (3), we have infinitely many solutions to pair of equations described by (2). Moreover, for $n \geq 1$ we notice that the subsequent solutions for both (5) and (6) beginning with $(x, y) = (0,1)$ for (5) and $(x, y) = (1,0)$ for (6) satisfy the recurrence relations $x_{n+2} = (k^2+2)x_{n+1} - x_n, y_{n+2} = (k^2+2)y_{n+1} - y_n$ (7).

4. Conclusion

After defining metallic ratios of order k, for any natural number k as in (1) and describing the pair of Diophantine equations as in (2), we have obtained the continued fraction expansion of the metallic ratio of order k as given in (3). Using the successive convergents of this continued fraction expansion, we have extracted the solutions in positive integers to the given pair of Diophantine equations. We notice that, since k is a positive integer all solutions obtained in (5) and (6) are also positive integers. Finally, a common recurrence relation for determining subsequent x and y values has been provided in (7). Thus, in this paper, we have completely solved the given pair of Diophantine equations in positive integers, using the successive convergents of continued fraction expansion of metallic ratio of order k. Using similar ideas, once can always try to generalize these concepts and solve other special types of Diophantine equations.

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